On Fluid Maxwell Equations

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A new formulation of equations of compressible fluids by analogy with Maxwell's equations

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Abstract
A compressible ideal fluid is governed by Euler's equation of motion and equations of continuity, entropy and vorticity. This system can be reformulated in a form analogous to that of electromagnetism governed by Maxwell's equations with source terms. The vorticity plays the role of magnetic field, while the velocity field plays the part of a vector potential and the enthalpy (of isentropic flows) plays the part of a scalar potential in electromagnetism. The evolution of source terms of fluid Maxwell equations is determined by solving the equations of motion and continuity. The equation of sound waves can be derived from this formulation, where time evolution of the sound source is determined by the equation of motion. The theory of vortex sound of aeroacoustics is included in this formulation. It is remarkable that the forces acting on a point mass moving in a velocity field of an inviscid fluid are analogous in their form to the electric force and Lorentz force in electromagnetism. The significance of the reformulation is interpreted by examples taken from fluid mechanics. This formulation can be extended to viscous fluids without difficulty. The Maxwell-type equations are unchanged by the viscosity effect, although the source terms have additional terms due to viscosities.

1. Introduction

A system of equations of compressible fluids is reconsidered from a new point of view. Namely, the system can be reformulated in a form analogous to that of electromagnetism governed by Maxwell's equations. This is motivated by a close similarity of structure of the equations of both systems, described below. From the present Maxwell-type formulation, one can immediately derive an equation of sound waves with source terms, whose time evolution

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is determined by the equations of motion and continuity. This is a new derivation of the basic equations of aeroacoustics.

In the past, several attempts have been made to make such a kind of Maxwell-type formalism of fluid systems. However, most were carried out under restricted conditions. For example, an analogy was presented between Maxwell's equations and fluid equations for an incompressible fluid (for the case of div $\boldsymbol{v} = 0$) by Troshkin (1993) first and by Marmanis (1998). The former was a study of turbulent fluctuations by a system of equations of average velocities and Reynolds stresses with an empirical closure formula. The field of perturbations is described by Maxwell's equations and Lorentz force in an ideal turbulent medium. On the other hand, Marmanis presented an analogy between the incompressible Navier–Stokes equations and Maxwell equations and applied it to the study of averaged field quantities of turbulence.

Another approach was proposed by Popov (1973), Ambegaokar *et al* (1980) and Arovas and Freire (1997) for the case of a two-dimensional superfluid. Any homogeneous superfluid described by a two-dimensional nonlinear Schrödinger-type energy functional is equivalent to $(2+1)$-dimensional electrodynamics, and equations analogous to the Maxwell equations are defined, where vortices play the role of charges. In analogy to electromagnetic radiation, sound radiation by the motion of vortices (termed as a vortex sound) in a superfluid was studied by Lundh and Ao (2000) and Parker *et al* (2004).

The present formulation is general and different from the above approaches, because the fluid is regarded as compressible and the flow field is three dimensional. The new formulation is presented in section 2. The wave equation of sound is derived quite naturally with this formulation (section 3). In addition, an equation of motion is derived in section 4 for a point mass moving in a fluid flow, where the force is given by a Coriolis term and a term like an electric force. The Coriolis term is equivalent to the Lorentz force in electromagnetism. This formulation can be extended to viscous fluids without difficulty. It is found that Maxwell-type equations are unchanged by the viscosity effect except for additional terms in the expression of source terms. Section 6 describes this finding.

In electromagnetism, Maxwell's equations for electric field $\boldsymbol{E}^{\mathrm{em}}$ and magnetic field $\boldsymbol{H}^{\mathrm{em}}$ are described by

$$\begin{aligned} &\nabla \times \boldsymbol{E}^{\mathrm{em}} + c^{-1}\partial_t \boldsymbol{H}^{\mathrm{em}} = 0, \quad \nabla \cdot \boldsymbol{H}^{\mathrm{em}} = 0, \\ &\nabla \times \boldsymbol{H}^{\mathrm{em}} - c^{-1}\partial_t \boldsymbol{E}^{\mathrm{em}} = \boldsymbol{J}^{\mathrm{e}}, \quad \nabla \cdot \boldsymbol{E}^{\mathrm{em}} = q^{\mathrm{e}}, \end{aligned} \tag{1}$$

where $q^{\mathrm{e}} = 4\pi\rho^{\mathrm{e}}$ and $\boldsymbol{J}^{\mathrm{e}} = (4\pi/c)\boldsymbol{j}^{\mathrm{e}}$ with ρ^{e} and $\boldsymbol{j}^{\mathrm{e}}$ being the charge density and current density vector, respectively, and c the light velocity. The vector fields $\boldsymbol{E}^{\mathrm{em}}$ and $\boldsymbol{H}^{\mathrm{em}}$ are defined in terms of a vector potential $\boldsymbol{A}$ and a scalar potential $\phi^{(\mathrm{e})}$ by

$$\boldsymbol{E}^{\mathrm{em}} = -c^{-1}\partial_t \boldsymbol{A} - \nabla\phi^{(\mathrm{e})}, \quad \boldsymbol{H}^{\mathrm{em}} = \nabla \times \boldsymbol{A}.$$

Using these definitions[1], the above Maxwell equations require that $\boldsymbol{A}$ and $\phi^{(\mathrm{e})}$ satisfy the following equations (Landau and Lifshitz 1975, chapter 8):

$$c^{-1}\partial_t\phi^{(\mathrm{e})} + \nabla \cdot \boldsymbol{A} = 0 \quad \text{(Lorentz condition)}, \tag{2}$$

$$(c^{-2}\partial_t^2 - \nabla^2)\boldsymbol{A} = \boldsymbol{J}^{\mathrm{e}}, \quad (c^{-2}\partial_t^2 - \nabla^2)\,\phi^{(\mathrm{e})} = q^{\mathrm{e}}. \tag{3}$$

[1] The vector potential and scalar potential were already used by Maxwell (1873), and the equation of an electromagnetic wave was derived by using the vector potential.

In fluid mechanics, the equation of motion of an ideal fluid is given by the Euler equation:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p, \tag{4}$$

supplemented with the following three equations:

$$\partial_t \rho + \boldsymbol{v}\cdot\nabla\rho + \rho\,\nabla\cdot\boldsymbol{v} = 0, \tag{5}$$

$$\partial_t s + \boldsymbol{v}\cdot\nabla s = 0, \tag{6}$$

$$\partial_t \boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = 0, \tag{7}$$

where ρ is the fluid density, s is the entropy per unit mass, p is the pressure, $\partial_t = \partial/\partial t$, and $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$. The following notations are used for vectors of position, velocity and space derivative as $\boldsymbol{x} = (x_i)$, $\boldsymbol{v} = (v_i)$ and $\nabla = (\partial_i)$ in the Cartesian frame ($\partial_i \equiv \partial/\partial x_i$). Equation (5) is the continuity equation, and (7) is the vorticity equation, whereas equation (6) is the entropy equation, which ensures that each fluid particle keeps its initial entropy (i.e. adiabatic).

If the initial entropy field is uniform with a value s_0, the fluid keeps the *isentropic* state $s = s_0$ everywhere and at any later time. In this case, we have $(1/\rho)\nabla p = \nabla h$ from thermodynamics where h is the enthalpy per unit mass.[2] In isentropic flows, an enthalpy variation Δh and a density variation $\Delta\rho$ are related by

$$\Delta h = \frac{1}{\rho}\,\Delta p = \frac{a^2}{\rho}\,\Delta\rho, \tag{8}$$

where $a^2 = (\partial p/\partial\rho)_s$ denotes the derivative with s fixed and a is the sound speed. From this, we have $\partial_t\rho = (\rho/a^2)\partial_t h$ and $\nabla\rho = (\rho/a^2)\nabla h$. Then the Euler equation (4) and the continuity equation (5) are rewritten as

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla h = 0, \tag{9}$$

$$\partial_t h + \boldsymbol{v}\cdot\nabla h + a^2\nabla\cdot\boldsymbol{v} = 0. \tag{10}$$

Linearizing (9) by neglecting $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$, and linearizing (10) by neglecting $\boldsymbol{v}\cdot\nabla h$ and replacing a with a constant value a_0, we have

$$\partial_t \boldsymbol{v} + \nabla h = 0, \qquad \partial_t h + a_0^2\,\nabla\cdot\boldsymbol{v} = 0. \tag{11}$$

Eliminating $\boldsymbol{v}$ from the two equations, we obtain $(\partial_t^2 - a_0^2\nabla^2)h = 0$. Using it, we obtain the wave equation for $\boldsymbol{v}$ as well. Thus, we have

$$(\partial_t^2 - a_0^2\nabla^2)\,\boldsymbol{v} = 0, \qquad (\partial_t^2 - a_0^2\nabla^2)h = 0. \tag{12}$$

It is remarkable that we have a close analogy between the two systems of fluid and electromagnetism. The wave equations (12) are the same as those of (3) in vacuum where $q^{\mathrm{e}} = 0$ and $\boldsymbol{J}^{\mathrm{e}} = 0$. In addition, the second equation of (11) is analogous to the Lorentz condition (2) by the correspondence $(\boldsymbol{A}, \phi^{(\mathrm{e})}, c) \leftrightarrow (a_0\boldsymbol{v},\ h, a_0)$. This implies the possibility of formulation of *fluid* Maxwell equations for which a vector potential is played by $\boldsymbol{v}$ (a constant a_0 is omitted in the subsequent formulation for simplicity) and a scalar potential is played by h. This is described in the next section.

The new formulation is interpreted by four examples in section 5. Section 7 considers the uniqueness of the transformation between the system of fluid equations and that of fluid

[2] From thermodynamics, we have $\mathrm{d}h = (1/\rho)\,\mathrm{d}p + T\,\mathrm{d}s$, where T is the temperature. For a barotropic fluid, ρ is defined as a function of pressure p only: i.e. $\rho = \rho(p)$. In this case, we can define a function $\Pi(p)$, such that $(1/\rho)\nabla p = \nabla\Pi$, where $\Pi \equiv \int^p \mathrm{d}p'/\rho(p')$, and h is replaced by Π.

Maxwell equations with source terms, in which the form of forces acting on a point mass moving in a fluid flow plays a crucial role.

2. Fluid Maxwell equations

To start with, it is assumed that the fluid is isentropic, i.e. $s = s_0$. We define two vector fields $\boldsymbol{E}$ and $\boldsymbol{H}$ by

$$\boldsymbol{E} \equiv -\partial_t \boldsymbol{v} - \nabla h, \quad \boldsymbol{H} \equiv \boldsymbol{\omega} = \nabla \times \boldsymbol{v}, \tag{13}$$

where the field variables $\boldsymbol{v}$ and h are analogous to the vector potential and scalar potential of electromagnetism. From these, *fluid* Maxwell equations are derived for an ideal fluid[3]:

$$\nabla \cdot \boldsymbol{H} = 0, \tag{14}$$

$$\nabla \cdot \boldsymbol{E} = q, \tag{15}$$

$$\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{H} = 0, \tag{16}$$

$$a_0^2 \nabla \times \boldsymbol{H} - \partial_t \boldsymbol{E} = \boldsymbol{J}, \tag{17}$$

where a_0 denotes the sound speed in a uniform state of fluid at rest, and

$$q = -\partial_t(\nabla \cdot \boldsymbol{v}) - \nabla^2 h, \quad \boldsymbol{J} = \partial_t^2 \boldsymbol{v} + \nabla \partial_t h + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}). \tag{18}$$

It is easily confirmed that the following equation is satisfied between q and $\boldsymbol{J}$:

$$\partial_t q + \mathrm{div}\, \boldsymbol{J} = 0. \tag{19}$$

This is required by taking divergence of (17) and operating ∂_t on (15), and by adding both equations. Due to the conservation form (19), the variable q may be termed a *source density*, while $\boldsymbol{J}$ may be called a *current vector*. It will be seen in section 3.2 that the term q is, in fact, a sound source due to vortex motion. From the definition of $\boldsymbol{J}$ of (18), it is clear that the current $\boldsymbol{J}$ vanishes in steady irrotational flows. The significance of q and $\boldsymbol{J}$ is considered in section 3, and in section 5 by examples.

The equation of motion (9) defines the vector $\boldsymbol{E}$ of (13) in terms of the velocity $\boldsymbol{v}$ only. Namely, $\boldsymbol{E}$ is given by

$$\boldsymbol{E} = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v} + \nabla(\tfrac{1}{2}v^2), \tag{20}$$

where the second equality is a vector identity. This leads to another expression of q in terms of $\boldsymbol{v}$:

$$q = \mathrm{div}[(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}] = \nabla \cdot (\boldsymbol{\omega} \times \boldsymbol{v}) + \nabla^2(\tfrac{1}{2}v^2). \tag{21}$$

By using (10), the vector $\boldsymbol{J}$ of (18) is expressed by

$$\boldsymbol{J} = \partial_t^2 \boldsymbol{v} - \nabla\left((\boldsymbol{v} \cdot \nabla)h + a^2 \nabla \cdot \boldsymbol{v}\right) + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}). \tag{22}$$

Time evolution of these source terms q and $\boldsymbol{J}$ must be determined by solving equations (9) and (10). However, the four equations (14)–(17) are valid without recourse to the equation of motion (9), as explained just now.

The system of equations (14)–(17) is derived as follows. First, (14) and (15) are obtained directly from the two definitions of (13), where q is given by (18) or (21). Equation (16) is

[3] If $\boldsymbol{H}$ is replaced by $\boldsymbol{H}^{\mathrm{em}}/a_0$, equations (14)–(17) reduce to (1) of electromagnetism (Landau and Lifshitz 1975, chapter 4) and a_0 corresponds to the light speed c.

nothing but an identity resulting from the defining expressions (13). Using (20) for $\boldsymbol{E}$ obtained from the equation of motion, equation (16) reduces to the vorticity equation (7). The remaining last equation (17) is obtained in the following way. First, we have $\partial_t \boldsymbol{E} = -\partial_t^2 \boldsymbol{v} - \nabla \partial_t h$ from (13), and note equation (10) for $\partial_t h$. Then, we obtain

$$- \partial_t \boldsymbol{E} - \partial_t^2 \boldsymbol{v} = \nabla \partial_t h = -\nabla \left(a^2 \nabla \cdot \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) h \right)$$
$$= -a^2 \nabla (\nabla \cdot \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}) \nabla a^2 - \nabla ((\boldsymbol{v} \cdot \nabla) h) . \tag{23}$$

Rearranging the above, we finally obtain (17) through transforming $\nabla \times \boldsymbol{H} = \nabla \times (\nabla \times \boldsymbol{v})$ by applying the following vector identity:

$$\nabla (\nabla \cdot \boldsymbol{v}) = \nabla \times (\nabla \times \boldsymbol{v}) + \nabla^2 \boldsymbol{v}, \tag{24}$$

where the vector $\boldsymbol{J}$ is given by (22). Note again that equation (9) is not used in the derivations of the four equations (14)–(17).

It is remarked that the two fields, $\boldsymbol{E} = (\boldsymbol{v} \cdot \nabla) \boldsymbol{v}$ and $\boldsymbol{H} = \nabla \times \boldsymbol{v}$, can be defined only in the continuous velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, because they depend on spatial derivatives of $\boldsymbol{v}$. The fields of $\boldsymbol{E}$ and $\boldsymbol{H}$ are not defined in a system of discrete particles since the velocity of each discrete particle is a function of time t only.

3. Equation of a sound wave

3.1. Wave equation

When we consider sound waves, it is assumed that a localized velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ (without uniform flow) is generated at a certain instant in an unbounded inviscid fluid of uniform density ρ_0 (otherwise at rest), in which the sound wave has a constant velocity $a_0 = [(\partial p / \partial \rho)_s (\rho_0)]^{1/2}$. The equation of a sound wave is derived as follows.

Differentiating (17) with respect to t and eliminating $\partial_t \boldsymbol{H}$ by using (16), we obtain

$$\partial_t^2 \boldsymbol{E} + a_0^2 \nabla \times (\nabla \times \boldsymbol{E}) = -\partial_t \boldsymbol{J}. \tag{25}$$

The second term on the left can be rewritten by using the identity (24), with $\boldsymbol{v}$ replaced by $\boldsymbol{E}$. Then equation (25) reduces to

$$(\partial_t^2 - a_0^2 \nabla^2)(\boldsymbol{E} + \partial_t \boldsymbol{v}) = -a_0^2 \nabla (\nabla \cdot \boldsymbol{E}) - \partial_t \hat{\boldsymbol{J}}, \tag{26}$$

$$\hat{\boldsymbol{J}} = \boldsymbol{J} - (\partial_t^2 - a_0^2 \nabla^2) \boldsymbol{v} \equiv a_0^2 \, \nabla \hat{Q}, \tag{27}$$

$$\hat{Q} = (1 - \hat{a}^2) \nabla \cdot \boldsymbol{v} - a_0^{-2} (\boldsymbol{v} \cdot \nabla) h, \tag{28}$$

where $\hat{a} = a / a_0$. From (13), we have $\boldsymbol{E} + \partial_t \boldsymbol{v} = -\nabla h$. Therefore, we can integrate (26) spatially, since all the terms are of the form of the gradient of scalar fields. Dividing (26) by $-a_0^2$ and integrating it, we obtain the following wave equation for the scalar variable $\tilde{h} = \tilde{p} / \rho$:

$$(a_0^{-2} \, \partial_t^2 - \nabla^2) \, \tilde{h} = S(\boldsymbol{x}, t), \tag{29}$$

$$S \equiv q + \partial_t \hat{Q}, \quad q = \nabla \cdot \boldsymbol{E}, \tag{30}$$

where $\tilde{h} = h - h_0$ and $\tilde{p} = p - p_0$ denote variations of enthalpy and pressure from uniform values h_0 and p_0. We have two terms in the source term S of the wave. It is noted that the second term $\partial_t \hat{Q}$ vanishes in the case of uniform density, since we should have $a = a_0$ (hence $\hat{a}^2 = 1$) and $\boldsymbol{v} \cdot \nabla h = 0$ if ρ is uniform, since $\boldsymbol{v} \cdot \nabla h = (a^2 / \rho)(\boldsymbol{v} \cdot \nabla) \rho$ by (8).

Using a representative value U of flow velocity, one can define a Mach number of the flow by $M = U/a_0$. Let $\hat{\tau}$ and l be a time scale and a length scale over which the fluid velocity undergoes significant changes. Then the source term $\partial_t \hat{Q}$ is of the order $O(M^2)$ of smallness for flows at a low M. This is shown for one-dimensional unsteady flow in section 5.1 below. The first term $\nabla \cdot \boldsymbol{E}$ can be transformed to an expression of the order $O(M)$ by the theory of vortex sound given below. This implies that the source of sound waves at low-M flows is dominated by the term $q = \nabla \cdot \boldsymbol{E}$.

In free space (without any solid body), the wave equation (29) can be transformed into an integral form for the acoustic pressure $\tilde{p} = \rho_0 \tilde{h}$ of the form

$$\tilde{p}(\boldsymbol{x}, t) = \rho_0 \int \mathrm{d}\tau \int G(\boldsymbol{x}, \boldsymbol{y}, t - \tau)\, S(\boldsymbol{y}, \tau)\, \mathrm{d}^3 \boldsymbol{y}, \tag{31}$$

where Green's function G satisfies $(a_0^{-2}\,\partial_t^2 - \nabla^2)\, G = \delta(\boldsymbol{x} - \boldsymbol{y})\,\delta(t - \tau)$. In free space, it is given by $G = (4\pi|\boldsymbol{\xi}|)^{-1}\delta(t - \tau - |\boldsymbol{\xi}|/a_0)$, with $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{y}$, where $\boldsymbol{y}$ and τ are the spatial and temporal coordinates for the source term $S(\boldsymbol{y}, \tau)$.

3.2. Vortex sound

The above formulation enables us to consider sound generation by vortex flows at low Mach numbers and derive the formula of vortex sound. We concentrate on the contribution from the first source term $q = \nabla \cdot \boldsymbol{E}$ only (supposed to be $O(M)$ and verified by the result given below). In localized flows due to compact distribution of vorticity $\boldsymbol{\omega}(\boldsymbol{x}, t)$, the velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ decays as $|\boldsymbol{x}|^{-3}$ when $|\boldsymbol{x}| \to \infty$ (Kambe 1986).

After substantial analyses (see Kambe and Minota 1983, Appendix for details), we obtain

$$\tilde{p}(\boldsymbol{x}, t) = \frac{\rho_0}{4\pi} \int \frac{\nabla_y \cdot \boldsymbol{L}(\boldsymbol{y}, t)}{|\boldsymbol{x} - \boldsymbol{y}|}\, \mathrm{d}^3 \boldsymbol{y}\Big|_{t_\mathrm{r}} + \frac{\rho_0}{2\pi c^2} \frac{1}{r} K''(t_\mathrm{r}) \tag{32}$$

$$= \frac{\rho_0}{4\pi} \frac{\partial}{\partial x_i} \int \frac{L_i(\boldsymbol{y}, t_\mathrm{r})}{|\boldsymbol{x} - \boldsymbol{y}|}\, \mathrm{d}^3 \boldsymbol{y} + \frac{\rho_0}{2\pi c^2} \frac{1}{r} K''(t_\mathrm{r}), \tag{33}$$

where $t_\mathrm{r} = t - |\boldsymbol{x} - \boldsymbol{y}|/a_0$ (retarded time), $\nabla_y = (\partial/\partial y_i)$, $\boldsymbol{L} = \boldsymbol{\omega} \times \boldsymbol{v}$, and $K(t) = \int \frac{1}{2} v^2 \mathrm{d}^3 \boldsymbol{x}$ is the total kinetic energy ($K'(t) = \mathrm{d}K/\mathrm{d}t$). The total energy K is conserved in inviscid fluids and hence the second term vanishes.[4]

The expression (32) without the second term implies the following:

$$\frac{1}{a_0^2}\,\partial_t^2 \tilde{p} - \nabla^2 \tilde{p} = \rho_0\, \nabla \cdot \boldsymbol{L} = \rho_0\, \mathrm{div}(\boldsymbol{\omega} \times \boldsymbol{v}). \tag{34}$$

This is called the *equation of vortex sound* (Powell 1964, Howe 1975). The first non-vanishing integral of (33) was cast into the following form by Möhring (1978):

$$\tilde{p}(\boldsymbol{x}, t) = -\frac{\rho}{12\pi a_0^2} \frac{x_i x_j}{r^3} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int y_i\, (\boldsymbol{y} \times \mathrm{curl}\, \boldsymbol{L})_j\, \mathrm{d}^3 \boldsymbol{y}\Big|_{t_\mathrm{r}}.$$

[4] In the far field as $|\boldsymbol{x}| \to \infty$, the factor $(\partial/\partial x_i) \int$ of (33) is replaced by $-a_0^{-1}\,(x_i/|\boldsymbol{x}|)\,(\partial/\partial t_\mathrm{r}) \int$, in order to obtain the leading term of $O(|\boldsymbol{x}|^{-1})$.

The factor curl $\boldsymbol{L} = \nabla \times (\boldsymbol{\omega} \times \boldsymbol{v})$ can be replaced by $-\partial_t \boldsymbol{\omega}$ from equation (7), and the integral becomes linear with respect to $\boldsymbol{\omega}$. Hence it is also linear with respect to $\boldsymbol{v}$ and is regarded as $O(M)$. Thus, it is found that the acoustic pressure in the far field (where $\rho = \rho_0$) is given by

$$\tilde{p}(\boldsymbol{x}, t) = \frac{\rho_0}{a_0^2} \frac{x_i x_j}{r^3} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^3 Q_{ij}(t_r), \qquad Q_{ij}(t) = \frac{1}{12\pi} \int_{\mathrm{D}_t} y_i (\boldsymbol{y} \times \boldsymbol{\omega}(\boldsymbol{y}, t))_j \, \mathrm{d}^3 \boldsymbol{y}. \tag{35}$$

This is the formula of the quadrupole component of vortex sound (Kambe and Minota 1981). Thus, the present formulation describes aeroacoustics in quite a natural way.

4. The Lorentz force and fluid electric field

Here we consider another example that will strengthen the analogy of fluid mechanics to electromagnetism. Suppose that there is a solid particle in a flow field $\boldsymbol{v}(t, \boldsymbol{x})$ of an inviscid fluid and that the particle is a point mass of mass m. Regarding this particle like a *test* particle in the flow field, we seek to obtain the law of motion of the particle, located at $\boldsymbol{x}_{\mathrm{p}}(t)$ and subjected to a conservative force $-\nabla\,(m\phi)$ of (force) potential ϕ, where $m\phi = m\phi_{\mathrm{g}} + (\Delta V)\,p$, i.e. the force consists of the gravity force $-\nabla(m\phi_{\mathrm{g}})$ and the pressure force $-(\nabla p)\,\Delta V$, where ΔV denotes the volume of the small particle. (See footnote 5.)

We define local velocity $\boldsymbol{u}(t)$ of the particle relative to the fluid velocity $\boldsymbol{v}$ at $\boldsymbol{x}_{\mathrm{p}}(t)$. It is assumed that the particle is small enough, so that the fluid velocity varies only slightly over distances of the particle dimension. According to the hydrodynamic theory (e.g. Landau and Lifshitz 1987, section 11) of fluid flow, when a solid particle moves through the fluid (at rest), the fluid energy induced by the particle motion of velocity $\boldsymbol{u} = (u_i)$ is expressed in the form $\frac{1}{2} m_{ik} u_i u_k$ by using the *induced mass* tensor m_{ik}. We consider the law of particle motion in unsteady rotational flow of a compressible fluid as follows. It is assumed that the flow field $\boldsymbol{v}(t, \boldsymbol{x})$ is not influenced by the position and velocity of the particle.

By the above definition and reasoning, the total particle velocity is $\boldsymbol{u} + \boldsymbol{v}$, where $\boldsymbol{x}_{\mathrm{p}}(t) = \boldsymbol{\xi}(t) + \boldsymbol{X}(t)$, $\boldsymbol{u} = \mathrm{d}\boldsymbol{\xi}/\mathrm{d}t = \dot{\boldsymbol{\xi}}$, and $\mathrm{d}\boldsymbol{X}/\mathrm{d}t$ is defined by $\boldsymbol{v}(t, \boldsymbol{x}_{\mathrm{p}}(t))$. The kinetic energy of the particle is $\frac{1}{2} m\,(\boldsymbol{u} + \boldsymbol{v})^2$. This means that the interaction with the non-uniform flow field $\boldsymbol{v}(t, \boldsymbol{x})$ is described by the term $m\boldsymbol{u} \cdot \boldsymbol{v}$, under the force potential $m\phi$. The fluid energy $\frac{1}{2} m_{ik} u_i u_k$ must be taken into account in the Lagrangian L describing the particle motion. Thus the Lagrangian L will take the form

$$\begin{aligned} L(t, \boldsymbol{\xi}, \boldsymbol{u}) &= \tfrac{1}{2} m\,(\boldsymbol{u} + \boldsymbol{v})^2 + \tfrac{1}{2} m_{jk} u_j u_k - m\phi \\ &= \tfrac{1}{2} m \boldsymbol{u}^2 + \tfrac{1}{2} m_{jk} u_j u_k + m\boldsymbol{v} \cdot \boldsymbol{u} - m\phi, \end{aligned} \tag{36}$$

where the term $\frac{1}{2} m\boldsymbol{v}^2$ is omitted in the last expression because it is not influenced by the position and velocity of the particle by assumption. Relative motion of the particle $\boldsymbol{\xi}(t)$ is described by Lagrange's equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\xi}_i}\right) = \frac{\partial L}{\partial \xi_i}, \qquad \text{where } \boldsymbol{\xi}(t) = (\xi_i), \qquad \boldsymbol{u}(t) = \dot{\boldsymbol{\xi}} = (\dot{\xi}_i),$$

with $\mathrm{d}/\mathrm{d}t = \partial_t + \boldsymbol{u} \cdot \nabla$ and $\partial/\partial \xi_i = \partial/\partial x_i = (\nabla)_i$. Using (36), we obtain

$$\frac{\partial L}{\partial \dot{\xi}_i} = \frac{\partial L}{\partial u_i} = m u_i(t) + m_{ik} u_k + m v_i(t, \boldsymbol{x}),$$

$$\frac{\partial L}{\partial \xi_i} = m \partial_i (\boldsymbol{u}(t) \cdot \boldsymbol{v}) - m \partial_i \phi, \qquad \text{where } \partial_i = \partial/\partial x_i.$$

The factor $\boldsymbol{u}(t)$ in the term $\partial_i(\boldsymbol{u}(t)\cdot\boldsymbol{v})$ behaves like a constant since ∂_i acts on a function of the space variable $\boldsymbol{x}$ only. Substituting these, Lagrange's equation reduces to

$$m\frac{\mathrm{d}u_i}{\mathrm{d}t}+m_{ik}\frac{\mathrm{d}u_k}{\mathrm{d}t}+m\frac{\partial v_i}{\partial t}+m(\boldsymbol{u}\cdot\nabla)v_i=m\partial_i(\boldsymbol{u}(t)\cdot\boldsymbol{v})-m\partial_i\phi. \tag{37}$$

The fourth term on the left and the first term on the right can be combined by the following vector identity:

$$\boldsymbol{u}\times(\nabla\times\boldsymbol{v})=\nabla(\boldsymbol{u}(t)\cdot\boldsymbol{v}(\boldsymbol{x}))-(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}.$$

Using this and introducing the *induced* fluid momentum $\boldsymbol{P}^{(\mathrm{f})}=(P_i^{(\mathrm{f})})$ by $P_i^{(\mathrm{f})}=m_{ik}u_k$, equation (37) can be written as

$$m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{P}^{(\mathrm{f})}}{\mathrm{d}t}=-m\partial_t\boldsymbol{v}+m\boldsymbol{u}\times(\nabla\times\boldsymbol{v})-m\nabla\phi. \tag{38}$$

The first term $-m\partial_t\boldsymbol{v}$ on the right is the inertia force due to *time* change of the background fluid velocity. The second term $m\boldsymbol{u}\times(\nabla\times\boldsymbol{v})$ describes the Coriolis force in the local frame of reference rotating with angular velocity $\frac{1}{2}\nabla\times\boldsymbol{v}$. The last term $-m\nabla\phi$ is the potential force consisting of the gravity potential $\phi_{\mathrm{g}}=gz$ of uniform gravity and pressure force, where $-\nabla(\phi_g)=(0,0,-g)=\boldsymbol{g}$ in the Cartesian frame with z the vertically upward coordinate.[5]

Equation (38) can be rewritten in an interesting form by defining the total momentum of the particle and fluid as $\boldsymbol{P}=m\boldsymbol{u}+\boldsymbol{P}^{(\mathrm{f})}$, and by eliminating $\partial_t\boldsymbol{v}$ and $\nabla\times\boldsymbol{v}$ by using $\boldsymbol{E}$ and $\boldsymbol{H}$ defined in (13). Then equation (38) reduces to

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}=m\boldsymbol{E}+m\boldsymbol{u}\times\boldsymbol{H}+m\nabla h-m\nabla\phi.$$

If $\rho_{\mathrm{p}}=\rho$, we have $\nabla\phi-\nabla h=\nabla\phi_{\mathrm{g}}$, since $\nabla h=\nabla p/\rho$, and $\nabla\phi=\nabla\phi_{\mathrm{g}}+(\Delta V/m)\nabla p=\nabla\phi_{\mathrm{g}}+\nabla p/\rho_{\mathrm{p}}$, according to footnote 5. Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}=m\boldsymbol{E}+m\boldsymbol{u}\times\boldsymbol{H}-m\nabla\phi_{\mathrm{g}}\quad(P_i=mu_i+m_{ik}u_k). \tag{39}$$

This is analogous to the equation of a charged particle in an electromagnetic field, since a point mass with a mass m and an electric charge e is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}_{\mathrm{p}}^{\mathrm{e}})=e\boldsymbol{E}^{\mathrm{em}}+(e/c)\,\boldsymbol{v}_{\mathrm{p}}\times\boldsymbol{H}^{\mathrm{em}}-m\nabla\Phi_{\mathrm{g}}, \tag{40}$$

where $\boldsymbol{v}_{\mathrm{p}}^{\mathrm{e}}$ is the velocity of a charged particle, c the light velocity, $\boldsymbol{E}^{\mathrm{em}}$ and $\boldsymbol{H}^{\mathrm{em}}$ the electric and magnetic fields and Φ_{g} the gravitational potential per unit mass.

Thus, it is seen that there is a similarity between the two equations, (39) and (40). In order to compare the two equations, the vector $\boldsymbol{H}$ of the fluid system should be replaced by $\boldsymbol{H}/a_0$, as noted in footnote 3 of section 2. By comparing the two terms on the right-hand side of both equations, it is seen that the *charge* of the fluid corresponding to the electric charge e is played by the mass m. It is remarkable that the Coriolis force $m\boldsymbol{u}\times\boldsymbol{H}$ is analogous to the Lorentz force $(e/c)\boldsymbol{v}_{\mathrm{p}}\times\boldsymbol{H}^{\mathrm{em}}$ in electromagnetism.

Thus, it may be said that the vector $\boldsymbol{H}$ is determined by the Coriolis force acting on the particle of total momentum $\boldsymbol{P}$ in the rotational flow, whereas the vector $\boldsymbol{E}$ is determined by the force acting on the particle after subtracting the Coriolis force and the gravity force.

[5] The pressure force $-(\nabla p)\,\Delta V$ is obtained from a surface integral of the pressure p over the particle surface and transforming it into a volume integral (Landau and Lifshitz 1987, section 2). In a liquid of density ρ at rest, we have $p(z)=p_0-\rho gz$ (where $p_0=p(0)$), and the force represents the buoyancy force of the particle of density $\rho_{\mathrm{p}}=m/\Delta V$, since $-\nabla(m\phi)=m\boldsymbol{g}-(\nabla p)\,\Delta V=(\rho_{\mathrm{p}}-\rho)(\Delta V)\,\boldsymbol{g}$. If $\rho_{\mathrm{p}}=\rho$, then $\nabla(m\phi)=0$, and the buoyancy force vanishes.

5. Examples

The fluid Maxwell equations and significance of the source terms are interpreted by three examples from fluid mechanics: (i) *unsteady potential* flow, (ii) *turbulent* flow of an *incompressible* fluid and (iii) 3D *steady potential* flow. A fourth example is given in section 6.2 for *viscous rotational* flow.

5.1. 1D unsteady potential flow

First, we consider an example of 1D unsteady gas flow in which all variables are functions of the time t and the coordinate x of the Cartesian frame (x, y, z). The flow velocity is assumed to have the x-component $u(x, t)$ only: $\boldsymbol{u} = (u(x, t), 0, 0)$, so that the velocity field is irrotational $\nabla \times \boldsymbol{u} = 0$, and described by a potential $\Phi(x, t)$: $\boldsymbol{v} = \nabla\Phi = (\partial_x \Phi, 0, 0)$. The x-component of Euler's equation (9) is

$$\partial_t u + u\partial_x u = -\partial_x h. \tag{41}$$

Obviously, this has an integral given by

$$\partial_t \Phi + \tfrac{1}{2}u^2 + h = C, \quad u = \partial_x \Phi, \tag{42}$$

where C is a constant (or may depend on t). The continuity equation (5) is

$$\partial_t \rho + u\partial_x \rho + \rho\partial_x u = 0$$

and the isentropic equation is $\partial_t s + u\partial_x s = 0$.

Since $\boldsymbol{\omega} = 0$, we have $\boldsymbol{H} = 0$, while $\boldsymbol{E} = (E, 0, 0)$ has only the x-component:

$$E(x, t) = -\partial_t u - \partial_x h = -\partial_x(\partial_t \Phi + h) = \partial_x(\tfrac{1}{2}u^2), \tag{43}$$

where the integral (42) is used to obtain the last expression. The source density q and current vector $\boldsymbol{J} = (J, 0, 0)$ are expressed as

$$q(x, t) = \partial_x E = -\partial_x^2(\partial_t \Phi + h) = \partial_x^2(\tfrac{1}{2}u^2),$$
$$J(x, t) = -\partial_t E = \partial_t \partial_x(\partial_t \Phi + h).$$

Obviously, $\partial_t q + \partial_x J = 0$ is satisfied. It is easily checked that all the Maxwell equations (14)–(17) are satisfied by the above expressions.

In this case, the source term $\nabla \cdot (\boldsymbol{\omega} \times \boldsymbol{v})$ of vortex sound vanishes. The source function $S(x, t)$ of the wave equation (29) includes all the terms neglected in the analysis of vortex sound of section 3.2. Equation (29) is $(a_0^{-2}\partial_t^2 - \nabla^2)\tilde{h} = S$, where

$$S(x, t) = q + \partial_t \hat{Q},$$
$$q = \tfrac{1}{2}(u^2)_{xx}, \quad \hat{Q} = (1 - \hat{a}^2)u_x + \frac{1}{a_0^2}\Big[\tfrac{1}{2}(u^2)_t + \tfrac{1}{3}(u^3)_x\Big], \tag{44}$$

and the symbol $(\cdot)_x$ denotes $\partial_x(\cdot)$.

In the adiabatic process, the equation of state is described by $p/\rho^\gamma =$ const for a polytropic gas with a constant ratio γ of specific heats. The sound speed is given by $a^2 = \mathrm{d}p/\mathrm{d}\rho|_s = \gamma p/\rho$. In a steady gas flow, a^2 is given as a function of u as $a^2 = a_0^2 - \tfrac{1}{2}(\gamma - 1)u^2$ (Landau and Lifshitz 1987, section 83). If this is substituted in a^2 of (44), we have the wave source S represented as a function of flow velocity u and its derivatives. It is found that $\hat{Q}$ is of the order $O(M^2)$, since we have $1 - \hat{a}^2 = \tfrac{1}{2}(\gamma - 1)M^2$, where $M = u/a_0$ and $\hat{a} = a/a_0$. The first term q of the source S is the second x derivative of $\tfrac{1}{2}u^2$, which results in the $O(M^2)$

term, because the $\partial^2/\partial x^2$ in the source leads to multiplication of the factor a_0^{-2} in the far-field expression of pressure (see footnote 4). Thus, it is found that the source of sound is $O(M^2)$ in this case.

5.2. Incompressible turbulent flow

So far, we have considered flow fields of a compressible fluid. However, there are *perturbation waves* in turbulent flows of an incompressible fluid (of uniform density ρ_0). Suppose that the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ is governed by the Euler equation and the divergence-free condition:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p_* = 0, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{45}$$

where $p_* = p/\rho_0$ is the specific pressure. Denoting average velocity and pressure by $\langle \boldsymbol{u} \rangle$ and $\langle p \rangle$, the corresponding fluctuation components are expressed by $\boldsymbol{u}'$ and p'. Thus, total velocity and pressure are

$$\boldsymbol{u} = \langle \boldsymbol{u} \rangle + \boldsymbol{u}', \qquad p = \langle p \rangle + p',$$

where $\langle \boldsymbol{u}' \rangle = 0$ and $\langle p' \rangle = 0$. Note that $\boldsymbol{u} \cdot \nabla u_i = \partial_j (u_j u_i)$ since $\partial_j u_j = 0$ and that

$$u_j u_i = \langle u_j \rangle \langle u_i \rangle + u'_j u'_i + \langle u_j \rangle\, u'_i + u'_j\, \langle u_i \rangle. \tag{46}$$

Taking the average, the third and fourth terms vanish, and the second term defines the (averaged) Reynolds stress:

$$\tau_{ji}(\boldsymbol{x}, t) = \langle u'_j(\boldsymbol{x}, t)\, u'_i(\boldsymbol{x}, t) \rangle.$$

Taking the average of the ith component of the first of (45) and taking the average of the second, we obtain

$$\partial_t \langle u_i \rangle + \langle \boldsymbol{u} \rangle \cdot \nabla \langle u_i \rangle + \partial_j \tau_{ji} + \partial_i \langle p \rangle = 0, \quad \nabla \cdot \langle \boldsymbol{u} \rangle = 0. \tag{47}$$

An equation for the Reynolds stress τ_{ik} can be written in the form

$$\partial_t \tau_{ik} + \langle \boldsymbol{u} \rangle \cdot \nabla \tau_{ik} + \tau_{is} \partial_s \langle u_k \rangle + \tau_{ks} \partial_s \langle u_i \rangle + h_{ik} = 0, \tag{48}$$

where $h_{ik} = \partial_s \langle u'_i u'_k u'_s + p' u_i \delta_{ks} + p' u_k \delta_{is} \rangle - \langle p' (\partial_i u'_k + \partial_k u'_i) \rangle$. In order to close the sequence of moment equations, the tensor h_{ik} is expressed by an empirical form that depends on second-order moments and models turbulent diffusion and relaxation.

By the averaging operation, *transverse* wave properties are introduced in the system of equations (47) and (48). Nonlinear terms govern the energy exchange between the average motion and turbulent fluctuations by the action of the Reynolds stresses τ_{ik}. The interaction between the two equations is realized by exchange oscillations of velocity and stress fields, transforming them into transverse waves of interleaved *electric* and *magnetic* components of the perturbation field. The wave property is shown just below by a simple analysis. By applying the laws of mechanics to randomly fluctuating fluids (Troshkin 1993), Maxwell's equation and the Lorentz force are derived, where the turbulent fluctuation u' is analogous to molecular fluctuation, and the average fluctuation velocity $\bar{u}' = \sqrt{\tau_{kk}/3}$ is analogous to the sound speed in a gas although the wave is transverse.

According to Troshkin (1993), consider two different average states $\langle \boldsymbol{u}^{(0)} \rangle\ (= 0)$ and $\langle \boldsymbol{u}^{(1)} \rangle$. In the former state, we assume $\tau_{ik}^{(0)} = c^2 \delta_{ik}$ in a homogeneous turbulence of zero average velocity, where $c^2 = \tau_{kk}^{(0)}/3$, i.e. $c = \bar{u}'^{(0)}$. We define differences of average velocities, pressures and Reynolds stresses by

$$\boldsymbol{\xi} = \langle \boldsymbol{u}^{(1)} \rangle, \quad \zeta = \langle p^{(1)} \rangle - \langle p^{(0)} \rangle, \quad \eta_{ik} = \tau_{ik}^{(1)} - \tau_{ik}^{(0)},$$

which are assumed to be so small that linear perturbation analysis is possible. As a result, the perturbation fields are described by

$$\partial_t \xi_i + \partial_k \eta_{ki} + \partial_i \zeta = 0, \tag{49}$$

$$\partial_t \eta_{ki} + c^2 (\partial_k \xi_i + \partial_i \xi_k) + f_{ki} = 0, \tag{50}$$

where the solenoidal property $\partial_k \xi_k = 0$ must be satisfied, and f_{ik} is modeled by a trace-less tensor ($f_{kk} = 0$) for a closure of the moment equations. As a simplifying example, neglecting the last terms in the above two equations, we obtain

$$\partial_t \xi_i + \partial_k \eta_{ki} = 0, \quad \partial_t \eta_{ki} + c^2 (\partial_i \xi_k + \partial_k \xi_i) = 0.$$

Taking the derivative of the second equation with respect to x_k (and summing with respect to k) and eliminating $\partial_k \eta_{ki}$ from both equations, we obtain the wave equation $\partial_t^2 \xi_i - c^2 \partial_k^2 \xi_i = 0$. Owing to the solenoidal condition $\partial_k \xi_k = 0$, the wave is transverse.

Maxwell's equations can be derived by defining a vector potential $\boldsymbol{A}$ and a scalar potential ϕ by $\boldsymbol{A} = c\boldsymbol{\xi}$ and $\phi = \zeta + \frac{1}{3}\chi$ (where $\chi = \eta_{kk}$). From these, we can define the vectors $\boldsymbol{E}^{\mathrm{t}}$ and $\boldsymbol{H}^{\mathrm{t}}$ by

$$\boldsymbol{E}^{\mathrm{t}} = -c^{-1}\partial_t \boldsymbol{A} - \nabla\phi, \quad \boldsymbol{H}^{\mathrm{t}} = \nabla \times \boldsymbol{A} = c\nabla \times \boldsymbol{\xi}. \tag{51}$$

The average equation of motion (49) leads to $E_i^{\mathrm{t}} = \partial_k \eta_{ki} - \partial_i(\frac{1}{3}\chi)$. From the definition (51), we obtain three of Maxwell equations:

$$c^{-1}\partial_t \boldsymbol{H}^{\mathrm{t}} + \nabla \times \boldsymbol{E}^{\mathrm{t}} = 0, \quad \nabla \cdot \boldsymbol{H}^{\mathrm{t}} = 0, \quad \nabla \cdot \boldsymbol{E}^{\mathrm{t}} = q^{\mathrm{t}},$$

where $c = \sqrt{\tau_{kk}^{(0)}/3}$ and $q^{\mathrm{t}} = \partial_i \partial_k \eta_{ki} - \partial_i^2(\frac{1}{3}\chi)$. The fourth is obtained as follows. Applying the operation $c^{-1}\partial_k$ to (50) and summing with respect to k, we obtain

$$\nabla \times \boldsymbol{H}^{\mathrm{t}} - c^{-1}\partial_t \boldsymbol{E}^{\mathrm{t}} = \boldsymbol{J}^{\mathrm{t}},$$

where $\boldsymbol{J}^{\mathrm{t}} = c^{-1}\partial_k f_{ki}$. Note that $\partial_t \chi = \partial_t \eta_{kk} = 0$ from (50) since $\partial_k \xi_k = 0$ and $f_{kk} = 0$.

It is remarkable that the Maxwell equations thus derived for incompressible turbulence are equivalent to those of (1).

5.3. 3D steady potential flow

For a steady potential flow, the equation of continuity and Euler's equation are

$$\nabla \cdot (\rho \boldsymbol{v}) = \rho \nabla \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \rho = 0, \quad (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla h = -\frac{a^2}{\rho}\nabla \rho. \tag{52}$$

Taking the scalar product of the second equation with $\boldsymbol{v}$ and eliminating $\boldsymbol{v} \cdot \nabla \rho$ by using the first one, we obtain

$$a^2 \nabla \cdot \boldsymbol{v} - \boldsymbol{v} \cdot (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = 0. \tag{53}$$

Thus the equations of motion are reduced to a single equation for velocity only if a^2 is expressed in terms of velocity. Introducing the velocity potential by $\boldsymbol{v} = \nabla\phi$, the above equation reduces to

$$(a^2 - \phi_x^2)\phi_{xx} + (a^2 - \phi_y^2)\phi_{yy} + (a^2 - \phi_z^2)\phi_{zz} - 2(\phi_x\phi_y\phi_{xy} + \phi_y\phi_z\phi_{yz} + \phi_z\phi_x\phi_{zx}) = 0. \tag{54}$$

In this case, we have

$$\boldsymbol{E} = \nabla(\tfrac{1}{2}v^2), \quad \boldsymbol{H} = 0, \quad q = \nabla \cdot \boldsymbol{E} = \nabla^2(\tfrac{1}{2}v^2), \quad \boldsymbol{J} = 0.$$

The two equations $\boldsymbol{H} = 0$ and $\boldsymbol{J} = 0$ are obvious from definitions (13) and (18). The first $\boldsymbol{E} = \nabla(\frac{1}{2}v^2)$ is obtained from (20). Naturally, all the Maxwell equations are satisfied by these expressions.

The vanishing of current $\boldsymbol{J} = 0$ is also derived from (53). In fact, from definitions (22), we obtain

$$\boldsymbol{J} = -\nabla\left[a^2\,\nabla\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\nabla h\right],$$

by the assumption of *steady* ($\partial_t^2\boldsymbol{v} = 0$) and *potential flow* ($\nabla\times\boldsymbol{v} = 0$). By using (52), we have $\boldsymbol{v}\cdot\nabla h = -\boldsymbol{v}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$.

Thus we obtain

$$\boldsymbol{J} = -\nabla\left[a^2\nabla\cdot\boldsymbol{v} - \boldsymbol{v}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = 0,$$

by (53). Equation (54) is investigated in detail for 2D flows in Landau and Lifshitz (1987, section 114).

6. Viscosity effect

6.1. General formulation

Viscous effects are taken into account in the present formulation without difficulty. Viscous stress tensor σ_{ik} is defined by

$$\sigma_{ik} = \mu e_{ik} + \zeta\,\delta_{ik}\mathcal{D},$$
$$e_{ik} = \partial_k v_i + \partial_i v_k - \tfrac{2}{3}\delta_{ik}\mathcal{D}, \qquad \mathcal{D} \equiv \partial_k v_k,$$

where μ and ζ are the coefficients of shear viscosity and bulk viscosity. Then the ith component of the equation of motion (9) is modified to the following form:

$$\partial_t v_i + (\boldsymbol{v}\cdot\nabla)v_i = -\partial_i h + Z_i, \tag{55}$$

$$Z_i \equiv T\,\partial_i s + \rho^{-1}\,\partial_k\sigma_{ik}, \tag{56}$$

where the pressure gradient term $-(1/\rho)\partial_i p$ is replaced by $-\partial_i h + T\,\partial_i s$, since $\mathrm{d}h = (1/\rho)\,\mathrm{d}p + T\,\mathrm{d}s$ from thermodynamics, T being the temperature. Equation (55) is the Navier–Stokes equation for a viscous compressible fluid.[6]

Equation (5) is unchanged. However, equations (6) and (7) are changed to

$$\partial_t s + \boldsymbol{v}\cdot\nabla s = D_*/\rho T, \quad D_* \equiv q_* + \mathrm{div}(k\nabla T), \quad q_* \equiv \tfrac{1}{2}\mu\, e_{ik}e_{ik} + \zeta\Delta^2, \tag{57}$$

$$\partial_t\boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = \boldsymbol{R}, \qquad \boldsymbol{R} \equiv \nabla\times\boldsymbol{Z}, \tag{58}$$

where q_* is the viscous rate of heating (Landau and Lifshitz, 1987, section 49), k the thermal conductivity, and $\boldsymbol{Z} = (Z_i)$ from (56).

Owing to the entropy production $D_*/\rho T$ of (57), we must use the thermodynamic relation $\mathrm{d}\rho = (\rho/a^2)\big(\mathrm{d}h - T\,\mathrm{d}s\big)$ instead of $\mathrm{d}\rho = (\rho/a^2)\mathrm{d}h$ in inviscid isentropic flows. Combining the continuity equation (5) and the entropy equation (57), we obtain the following equation, instead of (10):

$$\partial_t h + \boldsymbol{v}\cdot\nabla h + a^2\nabla\cdot\boldsymbol{v} = T(\partial_t s + \boldsymbol{v}\cdot\nabla s) = D_*/\rho. \tag{59}$$

[6] If the viscosity coefficients μ and ζ are constants, we have $\partial_k\sigma_{ik} = \mu\nabla^2\boldsymbol{v} + (\zeta + \frac{1}{3}\mu)\nabla\mathcal{D}$ and $\boldsymbol{R} = \nabla T\times\nabla s + \nu\nabla^2\boldsymbol{\omega}$. In an incompressible fluid of density ρ_0, we have $\mathcal{D} = 0$, and $\boldsymbol{Z} = T\nabla s + \nu\nabla^2\boldsymbol{v} = \nabla h - \rho_0^{-1}\nabla p + \nu\nabla^2\boldsymbol{v}$ (where $\nu = \mu/\rho_0$).

Using the original definition (13) for $\boldsymbol{H}$ and $\boldsymbol{E}$, we again obtain four Maxwell-type equations with the same forms as (14)–(17) for viscous fluids too. Namely, the Maxwell-type equations are not changed except the following. Owing to (55) and (56), we have

$$\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \boldsymbol{Z} = \boldsymbol{\omega}\times\boldsymbol{v} + \nabla(\tfrac{1}{2}v^2) - \boldsymbol{Z}. \tag{60}$$

Using this, the third equation (16) reduces to the viscous vorticity equation (58) for $\boldsymbol{\omega} = \boldsymbol{H}$. The vector $\boldsymbol{J}$ is given by (22), where $(\boldsymbol{v}\cdot\nabla)h$ of (22) should be replaced by $(\boldsymbol{v}\cdot\nabla)h - T(\partial_t s + \boldsymbol{v}\cdot\nabla s)$ and the rate of heating is given by $T(\partial_t s + \boldsymbol{v}\cdot\nabla s) = D_*/\rho$.

Thus it is found that the fluid Maxwell equations (14)–(17) are a fundamental system of equations, which is valid whether the fluid is inviscid or viscous.

The viscosity effect in the vortex sound is studied for collisions of two vortex rings by Kambe and Minota (1983) and Kambe (1984), experimentally and mathematically. In this case, the acoustic pressure in the far field has an additional monopole term,

$$\frac{5-3\gamma}{12}\frac{\rho_0}{\pi a_0^2}\frac{1}{r}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 K(t_r) \quad (\gamma : \text{specific heats ratio}),$$

in addition to the quadrupole term of (35). In an oblique collision of two vortex rings, reconnection of vortex lines occurs by the viscosity effect (Kambe *et al* 1993). Recent studies (Inoue *et al* 2000, Nakashima 2008) by direct numerical simulations revealed detailed physical processes of the wave generation of vortex sound by visualization.

6.2. Diffusing vortex (Lamb–Oseen vortex)

Let us consider an example of an axisymmetric diffusing vortex of a viscous compressible fluid with *circular* streamlines. In the cylindrical coordinates (r, θ, z), the velocity and vorticity are represented as

$$\boldsymbol{v}_* = (0,\ v(r,t),\ 0), \qquad \boldsymbol{\omega}_* = (0,\ 0,\ \omega(r,t)), \tag{61}$$

where $\omega = r^{-1}\partial(rv)/\partial r$. For all the field variables, the following derivatives vanish by the assumed symmetry: $\partial/\partial\theta = 0$, $\partial/\partial z = 0$, $\boldsymbol{v}_*\cdot\nabla = 0$. The velocity $\boldsymbol{v}_*$ of (61) has divergence-free property, $\nabla\cdot\boldsymbol{v}_* = 0$, so that the continuity equation reduces to $\partial_t\rho + \boldsymbol{v}_*\cdot\nabla\rho = \partial_t\rho = 0$. The density can be a function of r: $\rho_* = \rho(r)$.

Assuming uniform entropy of the fluid (as an initial test), the Navier–Stokes equation is written as $\partial_t\boldsymbol{v}_* + (\boldsymbol{v}_*\cdot\nabla)\boldsymbol{v}_* = -\rho^{-1}\nabla p + \nu\nabla^2\boldsymbol{v}_*$ (where $\nu = \mu/\rho$), whose radial and azimuthal components are given, respectively, by

$$-\frac{v^2}{r} = -\frac{1}{\rho}\partial_r p = -\partial_r h, \qquad \partial_t v = \nu\left(r^{-1}\partial_r(r\partial_r v) - \frac{v}{r^2}\right). \tag{62}$$

The vorticity equation reduces to

$$\partial_t\omega = \nu\nabla^2\omega, \qquad \nabla^2 = r^{-1}\partial_r(r\partial_r) = \partial_r^2 + r^{-1}\partial_r. \tag{63}$$

This has a solution for the initial condition $\omega(r,0) = \Gamma\,\delta(x)\delta(y)$. At a time $t > 0$, the z-component vorticity and the azimuthal velocity are given by

$$\omega_*(r,t) = \frac{\Gamma}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right), \qquad v_*(r,t) = \frac{\Gamma}{2\pi r}\left[1-\exp\left(-\frac{r^2}{4\nu t}\right)\right].$$

This is called the Lamb–Oseen vortex described in standard textbooks (e.g. Saffman 1992). With this solution, the vectors of fluid Maxwell equations are given by

$$\boldsymbol{H}_* = \boldsymbol{\omega}_* = (0, 0, \omega_*(r, t)), \quad \boldsymbol{E}_* = -\nu\nabla^2\boldsymbol{v}_* - \nabla h_*,$$

$$q_* = \mathrm{div}\boldsymbol{E} = -\nabla^2 h_*, \quad \boldsymbol{J}_* = (\nu\partial_t - a_0^2)\nabla^2\boldsymbol{v}_* + \partial_t\nabla h_*,$$

$$h_*(r, t) = \int^p \frac{\mathrm{d}p'}{\rho(p')} = \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{a^2}{\gamma-1}.$$

These are expressions derived from the solution. The source of sound in (29) is $S = \nabla\cdot\boldsymbol{E} + \partial_t\hat{Q}$, where $\hat{Q} = (1-\hat{a}^2)\nabla\cdot\boldsymbol{v} - a_0^{-2}(\boldsymbol{v}\cdot\nabla)h$. The term $\hat{Q}$ vanishes since $\nabla\cdot\boldsymbol{v}_* = 0$ and $\boldsymbol{v}_*\cdot\nabla h = 0$. Therefore, we have

$$S = \nabla\cdot\boldsymbol{E} = -\nabla^2 h_* = -r^{-1}\,\partial_r v_*^2,$$

from (62) together with the definition of ∇^2 of (63).

In a viscous fluid, however, there is heat release by viscous dissipation of kinetic energy (neglected in the above solution), which causes change of entropy and also change of density by thermal expansion. This results in non-vanishing divergence of velocity. This is considered below as a subsection.

There are two kinds of monopole terms in aerodynamic sound. One is related to a structure change of the velocity field (causing a change of the pressure field) due to local viscous diffusion, and the other is related to a density change due to local viscous heating (Kambe 1984). The above solution gives the former component of sound. The correction due to viscous heating considered below gives the latter component, which is opposite to the former.

Corrections by entropy production due to viscous dissipation

The rate of heat generation by viscosity is given by the local rate $Q_*(r, t) = \mu(e_{r\theta})^2$ per unit volume, where $e_{r\theta} = r\partial_r(v/r)$. This leads to an increase of local temperature by $\Delta T = Q_*\Delta t/(\rho c_\mathrm{p})$ during a time interval Δt, where c_p is the specific heat at constant pressure. This causes a density change by $\Delta\rho = -\rho\alpha\Delta T$, where α is the coefficient of thermal expansion. Rates of change of density and entropy are given by

$$\partial_t\rho = -\frac{\alpha}{c_p}Q_*, \quad \partial_t s = Q_*/\rho T.$$

The term $\partial_t\rho$ in the continuity equation induces a radial component of velocity $u(r, t)$ by the relation $\partial_t\rho + r^{-1}\,\partial_r(r\rho u) = 0$. Therefore, we have

$$u(r, t) = -\frac{1}{r\rho}\int^r r'\,\partial_t\rho\,\mathrm{d}r' = \frac{\alpha}{c_p\,r\rho}\int^r r'\,Q_*(r', t)\,\mathrm{d}r'.$$

Thus, the corrected state is described as

$$\boldsymbol{v} = (u(r, t), v(r, t), 0) = \boldsymbol{v}_* + \boldsymbol{v}', \quad \boldsymbol{v}' = (u(r, t), 0, 0)$$

$$h = h_*(r, t) + h'(r, t), \quad \rho = \rho_*(r, t) + \rho'(r, t),$$

where the variations of density and enthalpy are given by

$$\mathrm{d}\rho = -(\alpha Q_*/c_p)\,\mathrm{d}t, \quad \mathrm{d}h = (1/\rho)\,\mathrm{d}p + T\,\mathrm{d}s = (1/\rho)\,\mathrm{d}p + (Q_*/\rho)\,\mathrm{d}t.$$

The vectors of fluid Maxwell equations are given by $\boldsymbol{H} = \boldsymbol{H}_*$ and $\boldsymbol{E} = \boldsymbol{E}_* + \boldsymbol{E}'$, and

$$q = q_* + \mathrm{div}\,\boldsymbol{E}', \quad \boldsymbol{J} = \boldsymbol{J}_* - \partial_t\boldsymbol{E}', \quad \boldsymbol{E}' = -\partial_t\boldsymbol{v}' - \nabla h',$$

$$\mathrm{div}\,\boldsymbol{E}' = -\partial_t(\mathrm{div}\,\boldsymbol{v}') - r^{-1}\partial_r(r\partial_r h').$$

Now the source term q includes a contribution from non-zero $\partial_t \mathrm{div}\, \boldsymbol{v}'$, which is comparable to the previous one and could be a significant source of monopole sound.

7. Uniqueness

In section 2, the fluid Maxwell equations (14)–(17) have been derived from the system of fluid equations consisting of the Euler equation (9), continuity equation (5), entropy equation (6) and vorticity equation (7). For isentropic flows, the continuity equation (5) and entropy equation (6) are combined and represented by equation (10) for the enthalpy h. In addition, expressions of forces acting on a test particle in a flow field are given in section 4. It is found that the forces are analogous to the electric force and Lorentz force of electromagnetism under definition (13) of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ where $\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h$ and $\boldsymbol{H} = \nabla \times \boldsymbol{v}$. Moreover, the equation of motion (9) imposes the expression (20), i.e. $\boldsymbol{E} = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$, representing $\boldsymbol{E}$ in terms of $\boldsymbol{v}$ and its derivative only.

The subject of this section is how to recover the system of fluid equations (7), (9) and (10), for isentropic flows of an inviscid fluid, from the fluid Maxwell equations (14)–(17), which are supplemented with the source terms q and $\boldsymbol{J}$ given by (21) and (22), respectively. According to electromagnetism from a physical point of view and using it as a guiding principle in this analogous system, it is assumed that the forces acting on a test particle of mass m in flow field are given by the equation

$$\mathrm{d}\boldsymbol{P}/\mathrm{d}t = m\boldsymbol{E} + m\boldsymbol{u} \times \boldsymbol{H} - m\nabla\phi_{\mathrm{g}}, \tag{64}$$

for the rate of change of the total momentum $\boldsymbol{P}$. (This is equivalent to (39).) Comparing this with the equation (38) derived from the variational principle of mechanics in section 4, we obtain the following expressions:

$$\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h, \qquad \boldsymbol{H} = \nabla \times \boldsymbol{v}. \tag{65}$$

This is nothing but definition (13). Using these representations of $\boldsymbol{E}$ and $\boldsymbol{H}$, the first and third equations (14) and (16) of fluid Maxwell equations are satisfied identically. Hence, for the next step to deduce (7), (9) and (10), we start from the remaining second and fourth equations, (15) and (17), together with the source terms

$$q = \nabla \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}], \qquad \boldsymbol{J} = \partial_t^2 \boldsymbol{v} - \nabla[\boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v}] + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}), \tag{66}$$

defined by (21) and (22).

First we consider the fourth Maxwell equation (17), from which we try to derive equation (10) under the condition of no mass source. In fact, substituting the expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$ of (65) and for $\boldsymbol{J}$ of (66), the fourth Maxwell equation (17) is

$$\partial_t^2 \boldsymbol{v} + \nabla \partial_t h + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}) = \partial_t^2 \boldsymbol{v} - \nabla[\boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v}] + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}).$$

From this, we obtain $\nabla\big(\partial_t h + \boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v}\big) = 0$. Therefore,

$$\partial_t h + \boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v} = C(t) \quad \text{(a function of time } t\text{)}. \tag{67}$$

The right-hand side must vanish, i.e. $C(t) = 0$, in the absence of a mass source, because the left-hand side is transformed to $(a^2/\rho)\big(\partial_t \rho + \boldsymbol{v} \cdot \nabla\rho + \rho\nabla \cdot \boldsymbol{v}\big)$ by using $\mathrm{d}h = (a^2/\rho)\mathrm{d}\rho$ for isentropic flows. Namely, if $C(t) \neq 0$, the term $(\rho/a^2)C$ expresses a mass source. Therefore, $C(t)$ must vanish, and equation (10) is recovered. Under the isentropic condition, this is equivalent to the following continuity equation:

$$\partial_t \rho + \boldsymbol{v} \cdot \nabla\rho + \rho\nabla \cdot \boldsymbol{v} = \partial_t \rho + \nabla \cdot (\rho\boldsymbol{v}) = 0. \tag{68}$$

Next, substituting $q = \mathrm{div}[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]$ of (21) into equation (15) and integrating it, we obtain

$$\boldsymbol{E} = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\times\xi, \tag{69}$$

where ξ is an arbitrary vector field. Since $\boldsymbol{E}$ is defined by $\boldsymbol{E} = -\partial_t\boldsymbol{v} - \nabla h$ due to (65), this reduces to

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla h = -\nabla\times\xi. \tag{70}$$

By an energy consideration to be made just below, the vector field $\nabla\times\xi$ on the right-hand side must vanish, and we obtain the Euler equation. In fact, taking the scalar product with $\boldsymbol{v}$, we have

$$\partial_t\tfrac{1}{2}v^2 + (\boldsymbol{v}\cdot\nabla)\tfrac{1}{2}v^2 + \boldsymbol{v}\cdot\nabla h = -\boldsymbol{v}\cdot(\nabla\times\xi).$$

Mutiplying this by ρ and multiplying (68) by $\frac{1}{2}v^2$, and summing up both equations, we obtain

$$\partial_t(\tfrac{1}{2}\rho v^2) + \nabla\cdot(\rho\boldsymbol{v}\,\tfrac{1}{2}v^2) + \rho\boldsymbol{v}\cdot\nabla h = -\rho\boldsymbol{v}\cdot(\nabla\times\xi).$$

The total energy per unit volume of fluid is given by $\frac{1}{2}\rho v^2 + \rho\epsilon$, where ϵ is the internal energy per unit mass. The rate of change of the first term is given by the above, while the rate of change of the second term is given by thermodynamics as

$$\partial_t(\rho\epsilon) + h\nabla\cdot(\rho\boldsymbol{v}) = 0,$$

when the entropy is uniform (Landau and Lifshitz 1987, section 6). Summing up these two equations, we obtain

$$\partial_t(\tfrac{1}{2}\rho v^2 + \rho\epsilon) + \nabla\cdot\left[\rho\boldsymbol{v}(\tfrac{1}{2}v^2 + h)\right] = -\rho\boldsymbol{v}\cdot(\nabla\times\xi). \tag{71}$$

The law of energy conservation (Landau and Lifshitz 1987, section 6) requires that the right-hand side must vanish. Therefore, the term $\nabla\times\xi$ must be zero identically. Thus, equation (70) reduces to the Euler equation (9): $\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla h = 0$.

From (69) with $\nabla\times\xi = 0$, we recover the expression (20): $\boldsymbol{E} = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$. Substituting this expression into (16) but using the second expression of (20), and noting $\boldsymbol{H} = \boldsymbol{\omega}$, the third Maxwell equation (16) reduces to the vorticity equation (7).

Thus, under the conditions of the isentropy and energy conservation in the absence of mass source, the fluid equations (9), (10) and (7) are recovered from the definition of forces (64), Maxwell equations (14)–(17) and the source expressions (21) and (22).

In the case of a viscous fluid, the expression q must be replaced by $q = \mathrm{div}[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \boldsymbol{Z}]$ according to (60), where Z_i is defined by (56). From these, the Navier–Stokes equation (55) and the vorticity equation (58) are deduced. In (67), there must be an additional term $-T(\partial_t s + \boldsymbol{v}\cdot\nabla s)$. Then the same reasoning as that given just below (67) leads to the continuity equation (5). In the viscous case, the fluid Maxwell equations must be supplemented by (57).

8. Summary and discussions

Guided by the analogy to electromagnetism, the system of fluid equations is reformulated. The fluid system consists of Euler's equation of motion, continuity equation, entropy equation and vorticity equation. For isentropic flows, the continuity equation and entropy equation are combined and represented by a single equation of the enthalpy.

To begin with, two vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are defined in terms of the velocity vector $\boldsymbol{v}$ and enthalpy h, which are analogous to the vector potential and scalar potential of electromagnetism. Then, fluid Maxwell equations for $\boldsymbol{E}$ and $\boldsymbol{H}$ have been derived together with source terms, which are the source density q and current vector $\boldsymbol{J}$ satisfying the conservation form $\partial_t q + \text{div}\, \boldsymbol{J} = 0$. Evolution of the source terms is determined by solving the equations of motion and continuity. In this analogy, the field $\boldsymbol{H} = \nabla \times \boldsymbol{v}$ (vorticity) plays the role of magnetic field, while the field $\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h$ plays the role of electric field. Euler's equation of motion imposes the vector field $\boldsymbol{E}$ to be given by $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$. As a result, one of the Maxwell equations reduces to the vorticity equation.

Another analogy is the equation of motion of a test particle in the flow field, which takes a form analogous to the equation of motion of a charged particle in the electric field $\boldsymbol{E}^{\text{em}}$ and magnetic field $\boldsymbol{H}^{\text{em}}$. The Coriolis force acting on the test particle in the fluid flow is analogous to the Lorentz force in electromagnetism.

From the fluid Maxwell equations, one can derive quite naturally an equation of sound waves with source terms. Hence the sound wave is analogous to the electromagnetic wave in electromagnetism. This is a new derivation of the basic equations of aeroacoustics. The theory of vortex sound in aeroacoustics is included in this formulation. The wave source is dominated by a term of the form $q = \nabla \cdot \boldsymbol{E}$ at low Mach number flows, and the formulae of vortex sound are recovered.

It should be remarked that the two fields, $\boldsymbol{E} = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ and $\boldsymbol{H} = \nabla \times \boldsymbol{v}$, can be defined only in continuous fields, because they depend on spatial derivatives of velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. Those fields are not defined in discrete particle mechanics since the velocities of discrete particles are functions of time t only. The similarities found between electromagnetism and fluid mechanics imply the following. Just as the electromagnetic Maxwell equations are regarded as equations of gauge fields, the equations of continuity, entropy and vorticity of fluid flows are also regarded so (Kambe 2003, 2008a, 2008b). Electromagnetism is a system of Lorentz symmetry, whereas fluid flows are characterized by Galilean symmetry (Kambe 2007). Hence, both systems are different with respect to physical characterization. Even so, there is a similarity between them. This implies that the *existence* of symmetry is more fundamental than the *difference* of symmetries.

The present formulation can be extended to viscous fluids. In this extension, the fluid Maxwell equations are unchanged and valid whether the fluid is inviscid or viscous, although the source density and current vector have additional terms in a viscous fluid.

Uniqueness of transformation between the system of equations of fluid mechanics and the fluid Maxwell equations (together with source terms) is discussed in the last section, in which the form of forces acting on a point mass moving in a fluid flow played a crucial role. The significance of the transformation is interpreted by four examples taken from fluid mechanics.

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